The velocity with which the string leaves the pulley

$$=\sum_{x=1}^{x=n-1}\sqrt{\frac{2gl\,(2x-1)}{n(2n+1)}}=\sqrt{\frac{2gl}{n(2n+1)}}[1+\sqrt{3}+\sqrt{5}+\ldots.+\sqrt{(2n-3)}].$$

Pressure on pulley at time $t=\dfrac{4m^2(n+x+1)(n-x)}{m(2n+1)}=\dfrac{4m(n+x+1)(n-x)}{2n+1}$

DIOPHANTINE ANALYSIS.

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92. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the sides of integral right triangles when the difference of the legs is given.

Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Let a be the difference in the legs, and x and $x+a$=legs. Then $2x^2+2ax+a^2=\square=$(say) $[px-a]^2=p^2x^2-2apx+a^2$. Whence, $x=\dfrac{2a[p+1]}{p^2-2}$. Take $p=2$, $x=3a$, and the sides are $3a$, $4a$, $5a$. Then in the formula $\dfrac{2[r+s]}{r+2s}$, we have $r/s=\frac{2}{1}$, then $p=\frac{3}{2}$, and $x=20a$, and the sides $20a$, $21a$, and $29a$, and so on *ad infinitum.*

Remark on Problem 94 by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

In his solution of this question, Professor Zerr gives up his general demonstration a little prematurely. It is true that "For integral values of m, m^2+m+1 is not a square," but fractional values of m lead to integral values of a, b, and c. The value of m which makes the expression a square is $\dfrac{2p+1}{p^2-1}$ in which p may be any number except one. Take $p=2$, and $m=\frac{5}{3}$. Then $a=\frac{40}{9}$, $b=\frac{24}{9}$, and $c=\frac{15}{9}$. Any common multiple of a, b, and c makes the square root of $\sqrt{[a^2+b^2+c^2]}$ a square and as the denominator is a square, makes abc a square, as well as these particular values. So a, b, and c may be taken=40, 24, and 15, respectively. Then $abc\sqrt{[a^2+b^2+c^2]}=14400=49$, a square number. However, the question does not call for integral values, and I had solved the question as follows from the point at which Professor Zerr leaves it. Substituting the value of $m=\dfrac{2p+1}{p^2-1}$ in the values of a, b, and c, reducing to common denominator, $[p^2-1]^2$, we have $a=p[2p^2+5p+2]$, $b=p[p+2][p^2-1]$, and $c=2p^3=p^2-2p-1$, in which p may be any number except one. Hence, there is an indefinite number of rational triangles whose area is a square.

96. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

(*a*) Find the least three integral numbers such that the difference of every two of them shall be a square number; (*b*) find the least three square numbers such that the difference of every two af them shall be a square number.

Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

(a). Take a=one of the numbers, any then if we take x^2+a and y^2+a =the other numbers, and two of the conditions are met, and we have only to make x^2-y^2 (the difference between the other two) a square. But x^2-y^2 is the expression for one of the sides of a right angled triangle. Hence we may take $x=p^2+q^2$ and $y=p^2-q^2$, or $2pq$ and the three numbers will be $[p^2+q^2]^2+a$ and $[2pq]^2+a$ or $[p^2-q^2]^2+a$ and a, in which a, p, and q may be any numbers. To obtain the least three take $a=1, p=2$, and $q=1$, and the numbers are 1, 10 or 17, and 26, the least being 1, 10 and 26.

(b). The solution of the second part of this problem is on page 113, April number. The problem was incorrectly numbered 92. ED.

97. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Find a general expression for the radius of the sphere which, dropped in (or partly in) a right cone full of water, will displace the most water; the radius of the sphere, and the width, height and slant height of the cone to be rational integral numbers.

Solution by the PROPOSER.

Let ABC be a section of the cone passing through the axes, AB being the diameter of the cone, CD the axis, BC the slant height, and EB and EF radii of the sphere. Let $BD=a$, $BC=b$, and $CD=c$. Let $BE=BF=x$ and $DE=y$; then $BD=\sqrt{[x^2-y^2]}$; then $\dfrac{\pi[x+y]^3}{6}+\dfrac{\pi[x^2-y^2][x+y]}{2}$=contents of sphere within the cone which must be a maximum. Omitting constants, this reduces to $2x^3+3x^2y-y^3$=maximum....[1].

But $BC:BD::CE:FE$; or $b:a::c-y:x$, and $bx=ac-ay$....[2].

Differentiating, and reducing, $dy=-bdx/a$. Differentiating [1], substituting the value of dy and reducing, we have $2ax[x+y]-b[x^2-y^2]=0$. Dividing by $x+y$, we have $2ax-b[x-y]=0$....[3].

$x+y=0$ gives $x=-y$. Substituting this value in [1], the expression becomes zero. Hence, this value of x does not answer the conditions of the question. Substituting the value of y in [3] as found in [2], and reducing, we have

$$x=\frac{abc}{[b-a][b+2a]}.$$

As a, b, and c are sides of a right angled triangle, take $b=p^2+q^2$, $a=p^2-q^2$, and $c=2pq$. Then

$$x=\frac{p[p^2+q^2][p^2-q^2]}{q[3p^2-q^2]}.$$

For integrals, $x=p[p^2+q^2][p^2-q^2]$, $2a=2q[p^2-q^2][3p^2-q^2]$,

$b=q[p^2+q^2][3p^2-q^2]$, $c=2pq^2[3p^2-q^2]$,

in which p and q may be any integral numbers, p being greater than q.